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## *CALCULATION OF RADICALS.*

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FOR want of room I shall confine my self to the statement of theorems and rules.

1. *Theorem:* Let  $z$  and  $a$  be any positive numbers and  $k$  any positive integer; further let  $x = \sqrt[k]{z}$ . Form the expression

$$P = a^{n-1} + a^{n-2}x + a^{n-3}x^2 + \dots + ax^{n-2} + x^{n-1} \\ = \frac{a^n - z}{a - x};$$

develop the power  $P^k$  in ascending powers of  $x$ , and always substitute in this development  $z$  for  $x^n$ , then the expression for  $P^k$  will assume the form

$$P^k = A_{k,n-1} + A_{k,n-2}x + A_{k,n-3}x^2 + \dots + A_{k,0}x^{n-1} \dots \dots \dots (1)$$

Now the  $n$  successive ratios

$$\frac{A_{k,n-1}}{A_{k,n-2}}, \frac{A_{k,n-2}}{A_{k,n-3}}, \dots, \frac{A_{k,1}}{A_{k,0}} \text{ and } \frac{z \cdot A_{k,0}}{A_{k,n-1}}$$

are  $n$  different expressions for the real root of  $n^{\text{th}}$  degree of  $z$  with the same degree of approximation; or what amounts to the same, for sup. lim.  $k = \infty$

and superior limit  $k = \infty$   $\frac{A_{k+h}}{A_{k+h-m}} = \sqrt[m]{z^m}$ .

The calculated values of the quantities  $A$ , which may be called *components*, are as follows:

Let  $A_0, A_1, A_2, \dots$  be the binomial coefficients in the expansion of  $(1+x)^k$ , and  $B_0, B_1, B_2, \dots$  the positive binomial coefficients in the expansion of  $(1+x)^{-k}$ , that is

$$A_0 = 1, \quad A_1 = \frac{k}{1}, \quad A_2 = \frac{k \cdot k - 1}{1 \cdot 2}, \text{ etc.},$$

$$B_0 = 1, \quad B_1 = \frac{k}{1}, \quad B_2 = \frac{k \cdot k + 1}{1 \cdot 2}, \text{ etc.};$$

then the calculated values of the components are

$$A_{k,n-1} = a^{k(n-1)} + (A_0 B_n - A_1 B_0) a^{k(n-1)-n} z + (A_0 B_{2n} - A_1 B_n + A_2 B_0) \\ \times a^{k(n-1)-2n} z^2 + (A_0 B_{3n} - A_1 B_{2n} + A_2 B_n - A_3 B_0) a^{k(n-1)-3n} z^3 + \dots$$

The series for this component as well as for all others is finite, for the expansion of  $P^*$  shows that the highest power of  $a$  must be smaller than  $k$ .

The next component is

$$\begin{aligned}
 A_{k,n-2} = & B_1 a^{k(n-1)-1} + (A_0 B_{n+1} - A_1 B_1) a^{k(n-1)-(n+1)} z \\
 & + (A_0 B_{2n+1} - A_1 B_{n+1} + A_2 B_1) a^{k(n-1)-(2n-1)} z^2 + \dots \\
 & \vdots \quad \vdots \quad \vdots \\
 & \vdots \quad \vdots \quad \vdots \\
 A_{k,n-h-1} = & B_h a^{k(n-1)-h} + (A_0 B_{n+h} - A_1 B_h) a^{k(n-1)-(n+h)} z \\
 & + (A_0 B_{2n+h} - A_1 B_{n+h} + A_2 B_h) a^{k(n-1)-(2n+h)} z^2 + \dots
 \end{aligned}$$

where  $h$  denotes any of the numbers  $0, 1, 2, \dots, n-1$ ; and finally the last two components, and the most simple, are;

$$\begin{aligned}
 A_{k,1} = & B_{n-2} a^{k(n-1)-(n-2)} + (A_0 B_{2n-2} - A_1 B_{n-2}) a^{k(n-1)-(2n-2)} z \\
 & + (A_0 B_{3n-2} - A_1 B_{2n-2} + A_2 B_{n-2}) a^{k(n-1)-(3n-2)} z^2 + \dots \\
 A_{k,0} = & B_{n-1} a^{k(n-1)-(n-1)} + (A_0 B_{2n-1} - A_1 B_{n-1}) a^{k(n-1)-(2n-1)} z \\
 & + (A_0 B_{3n-1} - A_1 B_{2n-1} + A_2 B_{n-1}) a^{k(n-1)-(3n-1)} z^2 + \dots
 \end{aligned}$$

The successive formation of the above components for  $k = 2, 3, 4$ , etc. I will call *linear* algorithms, for the degree of approximation is proportional to the number  $k$ .

2. Specializing for  $k = 2$ , we obtain as components of second order;

$$\left. \begin{aligned}
 & 1. a^{2n-2} + (n-1)a^{n-2} z; \\
 & 2. a^{2n-3} + (n-2)a^{n-3} z; \\
 & \vdots \quad \vdots \\
 & h a^{2n-(h+1)} + (n-h)a^{n-(h+1)} z; \\
 & \vdots \quad \vdots \\
 & (n-2)a^{n+1} + 2.a^n z; \\
 & (n-1)a^n + 1.z; \\
 & n a^{n-1}.
 \end{aligned} \right\} \dots \dots \dots \quad (3)$$

The ratio of the last two components is the well known method of Newton:

$$a_1 = \frac{(n-1)a^n + z}{na^{n-1}},$$

which was reproduced by Mr. Evans in the *ANALYST* of January, 1876. Of all the  $n$  different values for  $\sqrt[n]{z}$ , furnished by the components of second order, one will be the best, independent of  $z$  and  $a$ , and this is the one where  $h = \frac{1}{2}(n-1)$ ; i. e.

$$a_1 = a \cdot \frac{(n-1)a^n + z(n+1)}{(n+1)a^n + z(n-1)}, \dots \dots \dots \quad (4)$$

where  $a$  denotes any convenient initial value, and  $a_1$  the corrected value for  $\sqrt[n]{z}$ .—The method under (4) is of third order, and reappears among the  $n$  methods for  $k = 3$ .

3. Specializing for  $k = 3$ , we obtain the components of third order:

$$\left. \begin{aligned}
 & 1.a^{3n-3} + \frac{n^2 + 3n - 2.2}{2} a^{2n-3} z + \frac{(n-1)(n-2)}{2} a^{n-3} z^2; \\
 & \frac{3}{1} a^{3n-4} + \frac{n^2 + 5n - 6.2}{2} a^{2n-4} z + \frac{(n-2)(n-3)}{2} a^{n-4} z^2; \\
 & \frac{3.4}{1.2} a^{3n-5} + \frac{n^2 + 7n - 12.2}{2} a^{2n-5} z + \frac{(n-3)(n-4)}{2} a^{n-5} z^2; \\
 & \vdots \quad \vdots \quad \vdots \\
 & \frac{h(h+1)}{2} a^{3n-(h+2)} + \frac{n^2 + (2h+1)n - h(h+1).2}{2} a^{2n-(h+2)} z \\
 & \quad + \frac{(n-h)(n-h-1)}{2} a^{n-(h+2)} z^2; \\
 & \vdots \quad \vdots \quad \vdots \\
 & \frac{(n-1)n}{2} a^{3n-(n+1)} + \frac{(n+1)n}{2} a^{2n-(n+1)} z; \\
 & \frac{(n+1)n}{2} a^{3n-(n+2)} + \frac{(n-1)n}{2} a^{2n-(n+2)} z.
 \end{aligned} \right\} \dots (5)$$

The last two components furnish the method (4) again after a slight reduction. The method under (4) seems to be the most practical of all, considering its simple form and rapidity of approximation. If the initial value ( $a$ ) has any number of correct decimals, the next corrected value has three times this number of correct decimals.

4. By fixing any value of  $k$  and repeating with any of the  $n$  possible methods the same process with the number  $k$ , we have  $n$  different algorithms of the order  $k$ . For  $k = 2$  we double with every step the number of correct decimals; for  $k = 3$  we multiply the number of correct decimals by 3, and so on:—Our general principle furnishes methods of any required degree of approximation.

If we would avoid raising to high powers, we have to prepare the given number  $z$  by proper multiplication so that its value is nearly unity. In this case 1 is a good initial value. Then form all the components of the second order and its  $n$  algorithms, and take the arithmetical mean of them. This value will multiply the number of correct decimals of the initial value by four.

A theorem still more general than the one here explained, and numerical examples, I am obliged to suppress here for want of room.

[Dr. Eggers writes under date of May 18th, “The case of a revolving ellipsoid of three unequal axes is treated of in Kirchhof’s Vorlesungen über Mathematische Physik, (Leipzig, editor Teubner, 1876.) Vorlesung 25; and by Dirichlet in Abhandlungen der Königlichen gesellschaft der Wissenschaften zu Göttingen, volume 8, 1860; and Rankine treats of it in London Philos. Transactions 1863, Part I, p. 227.”]